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Deviation operator and deviation equations over spaces with affine connections and metrics

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Abstract

The notion of deviation operator over spaces with affine connection (L_n-spaces) is introduced and its applications to deviation equations is considered. On the basis of a deviation identity, by means of sufficient or necessary and sufficient conditions, different deviation equations are obtained and considered. It is shown that the deviation equation for auto-parallel trajectories in L_n-spaces (geodesics in V_n-spaces) allows also other solutions than the well-known solutions for auto-parallel (geodesic) trajectories. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

In the last decades differential-geometric methods have found their well deserved places in theoretical physics and especially in theoretical gravitational physics. On the one hand, the models of the space–time have been extended from Minkowskian and (pseudo) Riemannian spaces without torsion (V_n-spaces) to the more sophisticated (pseudo) Riemannian spaces with torsion (U_n-spaces) and to spaces with affine connections and metrics [(L_n, g)- and $(\bar{L}_n, g)$-spaces] [1,2]. On the other hand, objections to such type of generalizations have been stated based on three major arguments:

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1. The equivalence principle is not valid in spaces more general than (pseudo) Riemannian spaces without torsion (let us recall that the equivalence principle is related to the fact that an affine connection can vanish at a point or over a curve in a V_n-space).
2. A Lorentz basis changes under a parallel transport in a (L_n, g)-space. This deformation (change of the length of the vectors and the angle between them) causes a deformation of the light cone and on these grounds the decomposition of the space–time in time-like and space-like regions is violated. The last corollary leads to the abuse of the law of causality which is one of the most well-founded concepts in temporary physics [3].
3. The role of the torsion (and nonmetricity) [specific for (L_n, g)- and $(\bar{L}_n, g)$-spaces] is ignorable (or at least torsion and nonmetricity are not measurable) in macro- and microphysics at the present time [4].

Recently, all three objections could be ignored on the basis of the considerations and the results obtained in the last years.

1. The first objection can be ignored on the grounds of the fact found by Iliev [5–11] and later by Hartley [12] that an affine connection can vanish under a special choice of the basic system at a point or over a curve in a space with affine connection and metrics. Therefore, every differentiable manifold with an affine connection and metrics can be used as a model of space–time in which the equivalence principle holds.
2. The second objection can be ignored on the grounds of the fact found by Manoff [13,14] that in (L_n, g)- and $(\bar{L}_n, g)$-spaces special types of transports (called Fermi–Walker transports) exist which do not deform a Lorentz basis. Therefore (on the analogy of the parallel transport in V_n-spaces), the law of causality is not abuse in (L_n, g)- and $(\bar{L}_n, g)$-spaces if instead of a parallel transport (used in a V_n-space) we use a Fermi–Walker transport in these type of spaces. Moreover, there also exist other types of transports (called conformal transports) under which a light cone does not deform [15,16].
3. The third objection could be ignored on the grounds of the mathematical fact that torsion and nonmetricity could compensate the influence of the curvature and external forces in a dynamical system. The recent physical theories are not constructed under taking into account this fact but there are no evidences that this will not be done in future more comprehensive theories of dynamical systems.

In this background, a question arises about applications of generalizations of well-constructed mathematical models in the Einstein theory of gravity (ETG) to theories in (L_n, g)- and $(\bar{L}_n, g)$-spaces. Such models, for instance, are deviation equations used as theoretical basis for construction of gravitational wave detectors in ETG. They can be generalized for (L_n, g)- and $(\bar{L}_n, g)$-spaces and are worth being investigated.

The main object by means of which we can generate and investigate deviation equations is a *deviation operator* acting on contravariant vector fields over (L_n, g)- and $(\bar{L}_n, g)$-spaces. This paper is devoted to its properties and possible applications.

In general relativity, as a basis for the theoretical scheme for gravitational wave detectors proposed by Weber (1958–1961) and discussed by many authors [17–21], the geodesic deviation equation (proposed by Levi-Civita in 1925 in a co-ordinate basis) [22] has been used in the form

$$\frac{\mathrm{D}^2\xi^i}{\mathrm{d}s^2} = R^i{}_{jkl}u^j u^k \xi^l, \qquad u^i{}_{;j}u^j = a^i = 0, \tag{1}$$

or in the index-free form $\nabla_u\nabla_u\xi = [R(u,\xi)]u$, $a = \nabla_u u = 0$. Its generalization for non-geodesic trajectories ($a \neq 0$) (proposed by Synge and Schild in 1956 in a co-ordinate basis) in the form

$$\frac{\mathrm{D}^2\xi^i}{\mathrm{d}s^2} = R^i{}_{jkl}u^j u^k \xi^l + a^i{}_{;j}\xi^j, \qquad a^i = u^i{}_{;j}u^j, \tag{2}$$

or in index-free form $\nabla_u\nabla_u\xi = [R(u,\xi)]u + \nabla_\xi a$ has also been used by Weber in a special form for construction of gravitational wave detectors of the type of massive cylinders reacting to periodical gravitational processes. The application of these equations in experiments for detecting gravitational waves turned the attention of many authors to the considerations and proposals for new deviation equations. Two types of prerequisites for obtaining such equations are usually used:

1. Physical interpretation of deviation equations as equations for the relative acceleration between particles, moving on trajectories in (pseudo) Riemannian spaces without torsion, considered as models of space–time in general theory of relativity, or in relativistic continuum media mechanics [23–32].
2. Mathematical models for obtaining deviation equations by means of (covariant) differential operators, acting on vector fields in spaces with affine connection and metric [(L_n, g)-spaces] [special case: (pseudo) Riemannian spaces with torsion (U_n-spaces) or V_n-spaces].

In both types of methods, problems arise connected with the physical interpretation of the quantities defined in the equations as well as with finding exact solutions of the proposed equations. At the same time, there are many tangential- and cross-points between these methods [33–35,37,38,43–52].

From mathematical point of view, many of the proposed deviation equations by different authors can be obtained from the s.c. generalized deviation identity (generalized deviation equation) in (L_n, g)-spaces [34,35]

$$\nabla_u\nabla_u\xi \equiv [R(u,\xi)]u + \nabla_a\xi + T(\xi,a) - \nabla_u[T(\xi,u)] + [\mathcal{L}\Gamma(\xi,u)]u, \tag{3}$$

or in a (co-ordinate or non-co-ordinate) basis

$$(\xi^i{}_{;j}u^j)_{;k}u^k \equiv R^i{}_{klj}u^k u^l \xi^j + \xi^i{}_{;j}a^j + T_{kl}{}^i\xi^k a^l - (T_{kl}{}^i\xi^k u^l)_{;j}u^j + \mathcal{L}_\xi \Gamma^i_{kl}u^k u^l, \tag{4}$$

where

$$a = \nabla_u u = u^i{}_{;j}u^j e_i = a^i e_i, \quad u \in T(M),$$
$$e_i = \partial_i = \frac{\partial}{\partial x^i} \text{ (in a co-ordinate basis)}, \quad u^i{}_{;j} = e_j u^i + \Gamma^i_{kj}u^k, \quad \Gamma^i_{kj} \neq \Gamma^i_{jk}. \tag{5}$$

The operator $R(\xi,u)$ is the curvature operator

$$R(\xi,u) = -R(u,\xi) = \nabla_\xi\nabla_u - \nabla_u\nabla_\xi - \nabla_{\mathcal{L}_\xi u} = [\nabla_\xi,\nabla_u] - \nabla_{[\xi,u]}, \quad \xi, u \in T(M), \tag{6}$$

The operator $\mathcal{L}\Gamma(\xi, u)$ is the deviation operator [34]

$$\mathcal{L}\Gamma(\xi, u) = \mathcal{L}_\xi \nabla_u - \nabla_u \mathcal{L}_\xi - \nabla_{\mathcal{L}_\xi u} = [\mathcal{L}_\xi, \nabla_u] - \nabla_{[\xi,u]}, \quad \xi, u \in T(M). \tag{7}$$

The contravariant vector $\mathcal{L}_\xi u$ is the Lie derivative of the contravariant vector field u along the contravariant vector field ξ,

$$\mathcal{L}_\xi u = [\xi, u] = \nabla_\xi u - \nabla_u \xi - T(\xi, u). \tag{8}$$

The contravariant vector $\nabla_u \xi$ is the covariant derivative of the vector field ξ along the vector field u, $T(\xi, u)$ is the torsion vector field

$$\begin{aligned} &T(\xi, u) = T_{kl}{}^{i} \xi^k u^l e_i, \qquad T_{kl}{}^{i} = \Gamma^i_{lk} - \Gamma^i_{kl} \ \text{(in a co-ordinate basis } \{\partial_i\}), \\ &T_{kl}{}^{i} = \Gamma^i_{lk} - \Gamma^i_{kl} - C_{kl}{}^{i} \ \text{(in a non-co-ordinate basis } \{e_i\}), \\ &\mathcal{L}_{e_k} e_l = [e_k, e_l] = C_{kl}{}^{i} e_i. \end{aligned} \tag{9}$$

Remark. Part of the construction of a deviation operator $\mathcal{L}\Gamma(\xi, u)$, related to the contravariant vector field ξ, has been used by Kobayashi and Nomizu [36] for the case when ξ appears as a generator of an infinitesimal affine transformation which preserves an affine connection.

In the present paper, the notion deviation operator is introduced over L_n-spaces and on its basis different deviation equations are obtained and considered. It is shown that all deviation equations have the same form and structure as deviation equations in the standard spaces with affine connection and metrics [(L_n, g)-spaces] as far as covariant structures (covariant affine connection [39], covariant metric) have not been introduced in their structure. The reason for that statement is trivial because of the fact that in the construction of the deviation equations in L_n-spaces only structures related to the affine connection can be used. The situation changes when covariant structures are incorporated in the construction of the deviation equations. In such case, the kinematic characteristics of vector fields have to be taken into account [40]. It is also shown that the auto-parallel deviation equation (geodesic deviation equation in V_n-spaces) admit other than the well-known solutions. The necessary and sufficient condition for the existence of the auto-parallel deviation equation and the deviation equation of Synge and Schild are also found. In Section 2 the notion of deviation operator and its properties are introduced and considered. In Section 3 different types of deviation equation are found on the basis of the deviation identity for a contravariant vector field ξ called deviation vector field. Section 4 comprises some concluding remarks.

2. Deviation operator

By means of the structure of the *curvature operator* $R(\xi, u) = \nabla_\xi \nabla_u - \nabla_u \nabla_\xi - \nabla_{\mathcal{L}_\xi u} = [\nabla_\xi, \nabla_u] - \nabla_{[\xi,u]}$, the commutator $[\nabla_w, R(\xi, u)]$, $w, \xi, u \in T(M)$, can be presented in the form

$$[\nabla_w, R(\xi, u)] = [\nabla_w, \mathcal{L}\Gamma(\xi, u)] + [\nabla_w, [\nabla_\xi, \nabla_u]] - [\nabla_w, [\mathcal{L}_\xi, \nabla_u]], \tag{10}$$

where

$$\mathcal{L}\Gamma(\xi, u) = \mathcal{L}_\xi \nabla_u - \nabla_u \mathcal{L}_\xi - \nabla_{\mathcal{L}_\xi u} = [\mathcal{L}_\xi, \nabla_u] - \nabla_{[\xi,u]}. \tag{11}$$

The operator $\mathcal{L}\Gamma(\xi, u)$ appears as a new operator, constructed by means of the Lie-differential operator $\mathcal{L}_\xi$ and the covariant differential operator ∇_u.

Definition 1. The operator $\mathcal{L}\Gamma(\xi, u)$ is called the *deviation operator*. It has the following properties:

1. $\mathcal{L}\Gamma(\xi, u) + \mathcal{L}\Gamma(u, \xi) = \mathcal{L}_\xi \nabla_u + \mathcal{L}_u \nabla_\xi - (\nabla_u \mathcal{L}_\xi + \nabla_\xi \mathcal{L}_u), \xi, u \in T(M)$.
   $\mathcal{L}\Gamma(\xi, u) - \mathcal{L}\Gamma(u, \xi) = \mathcal{L}_\xi \nabla_u + \nabla_\xi \mathcal{L}_u - (\nabla_u \mathcal{L}_\xi + \mathcal{L}_u \nabla_\xi) - 2\nabla_{\mathcal{L}_\xi u}$.
2. $\mathcal{L}\Gamma(f\xi, u) = \mathcal{L}_{f\xi} \nabla_u - \nabla_u \mathcal{L}_{f\xi} - \nabla_{\mathcal{L}_{f\xi} u}, f \in C^r(M)$.
3. $\mathcal{L}\Gamma(\xi, fu) = f\mathcal{L}\Gamma(\xi, u)$.
4. $\mathcal{L}\Gamma(\xi, u) = \mathcal{L}\Gamma(\xi, u^\alpha e_\alpha) = u^\alpha \mathcal{L}\Gamma(\xi, e_\alpha)$ (follows from 3).
   Remark. Greek indices are used instead of Latin indices for a non-co-ordinate basis.
5. $\mathcal{L}\Gamma(\xi, u + v) = \mathcal{L}\Gamma(\xi, u) + \mathcal{L}\Gamma(\xi, v), \xi, u, v \in T(M)$.
6. $\mathcal{L}\Gamma(\xi + \eta, u) = \mathcal{L}\Gamma(\xi, u) + \mathcal{L}\Gamma(\eta, u), \xi, \eta, u \in T(M)$.
7. $\mathcal{L}\Gamma(\alpha\xi, u) = \alpha\mathcal{L}\Gamma(\xi, u), \alpha \in F(R \text{ or } C)$.
   From 3, 5–7, it follows that $\mathcal{L}\Gamma(\xi, u)$ appears as a bilinear operator with respect to the contravariant vector fields ξ and u.
8. $[\mathcal{L}\Gamma(f\xi, u)]v = f[\mathcal{L}\Gamma(\xi, u)]v + [(uv)f - (\nabla_u v)f]\xi + (uf)(\nabla_\xi - \mathcal{L}_\xi)v + (vf)\nabla_u \xi$, $f \in C^r(M), r \geq 2, \xi, u, v \in T(M)$.
9. $\mathcal{L}\Gamma(\xi, \xi) = [\mathcal{L}_\xi, \nabla_\xi]$, [compare with $R(\xi, \xi) = 0$].
10. Action of the deviation operator on a function f: $[\mathcal{L}\Gamma(\xi, u)]f = 0, f \in C^r(M), r \geq 2$.
11. Action of the deviation operator on a contravariant vector field:
    $[\mathcal{L}\Gamma(\xi, u)](fv) = f[\mathcal{L}\Gamma(\xi, u)]v, \xi, u, v \in T(M)$,
    $[\mathcal{L}\Gamma(\xi, u)]v = v^\beta[\mathcal{L}\Gamma(\xi, u)]e_\beta = v^j[\mathcal{L}\Gamma(\xi, u)]\partial_j = u^\gamma v^\beta[\mathcal{L}\Gamma(\xi, e_\gamma)]e_\beta$
    $= u^j v^i[\mathcal{L}\Gamma(\xi, \partial_j)]\partial_i$.
    The connections between the action of the deviation operator and that of the curvature operator on a contravariant vector field can be given in the form

$$[\mathcal{L}\Gamma(\xi, u)]v=[R(\xi, u)]v + [\nabla_u \nabla_v - \nabla_{\nabla_u v}]\xi - T(\xi, \nabla_u v) + \nabla_u[T(\xi, v)]. \tag{12}$$

In a co-ordinate basis, $[\mathcal{L}\Gamma(\xi, \partial_l)]\partial_k$ has the form

$$[\mathcal{L}\Gamma(\xi, \partial_l)]\partial_k = [\xi^i{}_{;k;l} - R^i{}_{klj}\xi^j + (T_{jk}{}^i \xi^j)_{;l}]\partial_i = (\mathcal{L}_\xi \Gamma^i_{kl})\partial_i. \tag{13}$$

The components $\mathcal{L}_\xi \Gamma^i_{kl}$ are called *Lie derivative of the components Γ^i_{kl} of a contravariant affine connection Γ* along the contravariant vector field ξ. It can be written also in the form [41]

$$\mathcal{L}_\xi \Gamma^i_{kl} = \xi^i{}_{,k,l} + \xi^j \Gamma^i_{kl,j} - \xi^i{}_{,j} \Gamma^j_{kl} + \xi^j{}_{,k} \Gamma^i_{jl} + \xi^j{}_{,l} \Gamma^i_{kj}. \tag{14}$$

By means of $\mathcal{L}_\xi \Gamma^i_{kl}$, the expression for $[\mathcal{L}\Gamma(\xi, u)]v$ can be presented in the form

$$\begin{aligned}[\mathcal{L}\Gamma(\xi, u)]v &= v^k u^l (\mathcal{L}_\xi \Gamma^i_{kl})\partial_i \\ &= [\xi^i{}_{;k;l} v^k u^l - R^i{}_{klj} v^k u^l \xi^j + (T_{jk}{}^i \xi^j)_{;l} v^k u^l]\partial_i.\end{aligned} \tag{15}$$

In this way, the second covariant derivative $\nabla_u \nabla_v \xi$ of the contravariant vector field ξ can be presented by means of the deviation operator in the form

$$\begin{aligned}\nabla_u \nabla_v \xi &= [R(u, \xi)]v + \nabla_\xi \nabla_u v - \mathcal{L}_\xi(\nabla_u v) - \nabla_u[T(\xi, v)] + [\mathcal{L}\Gamma(\xi, u)]v \\ &= ([R(u, \xi)]v) + \nabla_\xi \nabla_u v - \nabla_u \mathcal{L}_\xi v - \nabla_{\mathcal{L}_\xi u} v - \nabla_u[T(\xi, v)].\end{aligned} \tag{16}$$

For $v = u$, the last identity is called *generalized deviation identity* [34].

12. Action of the deviation operator on a contravariant tensor field $V = V^A e_A = V^{\alpha_1 \cdots \alpha_l} e_{\alpha_1} \otimes \cdots \otimes_{\alpha_l} \in \otimes^l(M)$:

$$\begin{aligned}[\mathcal{L}\Gamma(\xi, u)]V &= u^\gamma V^A [\mathcal{L}\Gamma(\xi, e_\gamma)] e_A = u^\gamma V^B (\mathcal{L}_\xi \Gamma^A_{B\gamma}) e_A \\ &= -(S_{B\alpha}{}^{A\beta} V^B \mathcal{L}_\xi \Gamma^\alpha_{\beta\gamma} u^\gamma) e_A,\end{aligned} \tag{17}$$

where

$$\begin{aligned}&\mathcal{L}_\xi \Gamma^A_{B\gamma} = -S_{B\alpha}{}^{A\beta} \mathcal{L}_\xi \Gamma^\alpha_{\beta\gamma}, \\ &[\mathcal{L}\Gamma(\xi, u)] e_B = (\mathcal{L}_\xi \Gamma^A_{B\gamma}) u^\gamma e_A \\ &\qquad = -S_{B\alpha}{}^{A\beta} [\xi^\alpha{}_{/\beta/\gamma} - R^\alpha{}_{\beta\gamma\delta} \xi^\delta + (T_{\delta\beta}{}^\alpha \xi^\delta)_{/\gamma}] u^\gamma e_A, \\ &S_{B\alpha}{}^{A\beta} = -\sum_{k=1}^{l} g^\beta_{\alpha_k} g^{\beta_k}_\alpha g^{\beta_1}_{\alpha_1} g^{\beta_2}_{\alpha_2} \cdots g^{\beta_{k-1}}_{\alpha_{k-1}} g^{\beta_{k+1}}_{\alpha_{k+1}} \cdots g^{\beta_l}_{\alpha_l}.\end{aligned} \tag{18}$$

13. The deviation operator obeys the Leibnitz rule for differentiation of tensor fields: $[\mathcal{L}\Gamma(\xi, u)](V \otimes S) = [\mathcal{L}\Gamma(\xi, u)]V \otimes S + V \otimes [\mathcal{L}\Gamma(\xi, u)]S$, $V \in \otimes^l(M)$, $S \in \otimes_k(M)$.
14. The deviation operator appears as linear differential operator acting on contravariant or covariant tensor fields:
 $[\mathcal{L}\Gamma(\xi, u)](\alpha V_1 + \beta V_2) = \alpha[\mathcal{L}\Gamma(\xi, u)]V_1 + \beta[\mathcal{L}\Gamma(\xi, u)]V_2$, $\alpha, \beta \in F(R \text{ or } C)$, $V_i \in \otimes^l(M)$, $i = 1, 2$,
 $[\mathcal{L}\Gamma(\xi, u)](\alpha W_1 + \beta W_2) = \alpha[\mathcal{L}\Gamma(\xi, u)]W_1 + \beta[\mathcal{L}\Gamma(\xi, u)]W_2$, $W_i \in \otimes_k(M)$, $i = 1, 2$.
15. The deviation operator obeys identity analogous to the first type Bianchi identity for the curvature operator

$$\begin{aligned}&\langle[\mathcal{L}\Gamma(\xi, u)]v\rangle \equiv \langle(\nabla_\xi \nabla_u - \nabla_{\nabla_\xi u})v\rangle + \langle T(T(\xi, u), v)\rangle - \langle T(u, \nabla_\xi v)\rangle, \\ &\xi, u, v \in T(M),\end{aligned} \tag{19}$$

where

$$\begin{aligned}&\langle[\mathcal{L}\Gamma(\xi, u)]v\rangle = [\mathcal{L}\Gamma(\xi, u)]v + [\mathcal{L}\Gamma(v, \xi)]u + [\mathcal{L}\Gamma(u, v)]\xi, \\ &\langle(\nabla_\xi \nabla_u - \nabla_{\nabla_\xi u})v\rangle = (\nabla_\xi \nabla_u - \nabla_{\nabla_\xi u})v + (\nabla_v \nabla_\xi - \nabla_{\nabla_v \xi})u + (\nabla_u \nabla_v - \nabla_{\nabla_u v})\xi, \\ &\langle T(u, \nabla_\xi v)\rangle = T(u, \nabla_\xi v) + T(v, \nabla_u \xi) + T(\xi, \nabla_v u).\end{aligned} \tag{20}$$

In a non-co-ordinate basis (where $_{/\alpha}$ means covariant derivative along e_α) this identity obtains the form

$$(\mathcal{L}_\xi \Gamma^\gamma_{\alpha\beta})v^\alpha u^\beta + (\mathcal{L}_u \Gamma^\gamma_{\alpha\beta})\xi^\alpha v^\beta + (\mathcal{L}_v \Gamma^\gamma_{\alpha\beta})u^\alpha \xi^\beta$$
$$\equiv \xi^\gamma{}_{/\alpha/\beta} v^\alpha u^\beta + u^\gamma{}_{/\alpha/\beta}\xi^\alpha v^\beta + v^\gamma{}_{/\alpha/\beta} u^\alpha \xi^\beta + T_{\langle\alpha\beta}{}^\kappa T_{\kappa\delta\rangle}{}^\gamma v^\alpha \xi^\beta u^\delta$$
$$-T_{\alpha\beta}{}^\gamma (u^\alpha v^\beta{}_{/\delta}\xi^\delta + v^\alpha \xi^\beta{}_{/\delta} u^\delta + \xi^\alpha u^\beta{}_{/\delta} v^\delta). \quad (21)$$

The commutator of the covariant differential operator and the deviation operator obeys the following identity:

$$\langle[\nabla_w, \mathcal{L}\Gamma(\xi, u)]\rangle \equiv \langle[\nabla_w, [\mathcal{L}_\xi, \nabla_u]]\rangle - \langle R(w, \mathcal{L}_\xi u)\rangle, \quad (22)$$

where

$$\langle[\nabla_w, \mathcal{L}\Gamma(\xi, u)]\rangle = [\nabla_w, \mathcal{L}\Gamma(\xi, u)] + [\nabla_u, \mathcal{L}\Gamma(w, \xi)] + [\nabla_\xi, \mathcal{L}\Gamma(u, w)],$$
$$\langle[\nabla_w, [\mathcal{L}_\xi, \nabla_u]]\rangle = [\nabla_w, [\mathcal{L}_\xi, \nabla_u]] + [\nabla_u, [\mathcal{L}_w, \nabla_\xi]] + [\nabla_\xi, [\mathcal{L}_u, \nabla_w]],$$
$$\langle R(w, \mathcal{L}_\xi u)\rangle = R(w, \mathcal{L}_\xi u) + R(u, \mathcal{L}_w \xi) + R(\xi, \mathcal{L}_u w), \qquad \xi, u, w \in T(M).$$

3. Deviation equations

By means of the explicit form of the deviation operator $\mathcal{L}\Gamma(\xi, u)$ and its connection with the curvature operator, the generalized deviation identity

$$\nabla_u \nabla_u \xi \equiv [R(u, \xi)]u + \nabla_a \xi + T(\xi, a) - \nabla_u[T(\xi, u)] + [\mathcal{L}\Gamma(\xi, u)]u, \qquad \nabla_u u = a, \quad (23)$$

can be presented in different equivalent forms:

$$\text{(a)} \quad \nabla_u \nabla_u \xi \equiv [R(u, \xi)]u + \nabla_\xi a - \mathcal{L}_\xi a - \nabla_u[T(\xi, u)] + [\mathcal{L}\Gamma(\xi, u)]u. \quad (24)$$

For obtaining this form, the relation $\mathcal{L}_\xi a = \nabla_\xi a - \nabla_a \xi - T(\xi, a)$ has been used. If the explicit form of $[\mathcal{L}\Gamma(\xi, u)]$ is substituted in the deviation identity, it will have the form

$$\text{(b)} \quad \nabla_u \nabla_u \xi \equiv [R(u, \xi)]u + \nabla_\xi a - \nabla_u[\mathcal{L}_\xi u + T(\xi, u)] - \nabla_{\mathcal{L}_\xi u} u, \quad (25)$$

or the form

$$\text{(c)} \quad \nabla_u \nabla_u \xi \equiv [R(u, \xi)]u + \nabla_a \xi + \mathcal{L}_\xi a + T(\xi, a) - \nabla_u[\mathcal{L}_\xi u + T(\xi, u)] - \nabla_{\mathcal{L}_\xi u} u. \quad (26)$$

In a co-ordinate basis, the generalized deviation identity can be written in the forms:

$$(\xi^i{}_{;j} u^j)_{;k} u^k \equiv R^i{}_{klj} u^k u^l \xi^j + \xi^i{}_{;j} a^j - (T_{kl}{}^i \xi^k)_{;j} u^j u^l + u^k u^l \mathcal{L}_\xi \Gamma^i_{kl}$$
$$= R^i{}_{klj} u^k u^l \xi^j + a^i{}_{;j}\xi^j - \mathcal{L}_\xi a^i - (T_{kl}{}^i \xi^k u^l)_{;j} u^j + u^k u^l \mathcal{L}_\xi \Gamma^i_{kl}$$
$$= R^i{}_{klj} u^k u^l \xi^j + a^i{}_{;j}\xi^j - (\mathcal{L}_\xi u^i)_{;j} u^j - u^i{}_{;j}(\mathcal{L}_\xi u^j) - (T_{kl}{}^i \xi^k u^l)_{;j} u^j$$
$$= R^i{}_{klj} u^k u^l \xi^j + a^i{}_{;j}\xi^j - u^i{}_{;j}(\mathcal{L}_\xi u^j) - (\mathcal{L}_\xi u^i + T_{kl}{}^i \xi^k u^l)_{;j} u^j, \quad (27)$$

where

$$u^k u^l \mathcal{L}_\xi \Gamma^i_{kl} = \mathcal{L}_\xi a^i - (\mathcal{L}_\xi u^i)_{;j} u^j - u^i{}_{;j}(\mathcal{L}_\xi u^j),$$
$$\mathcal{L}_\xi u^i = u^i{}_{;k}\xi^k - u^k \xi^i{}_{;k} - T_{kl}{}^i \xi^k u^l. \tag{28}$$

In analogous way the generalized deviation equation can be presented in a non-co-ordinate basis.

The generalized deviation equation in a co-ordinate basis has been used for obtaining different types of deviation equations in V_n-spaces, admitting certain types of symmetries [34]. It was found that deviation equations and their solutions proposed by different authors and obtained by different methods can be found and considered on the basis of unique method connected with the generalized deviation identity and some additional conditions for the contravariant vector fields u and ξ. These conditions are generally covariant equations of first order for u or ξ. If such conditions determine the contravariant vector field ξ or are given as covariant equations of first order for u or ξ (or only for ξ) (for example, conditions of the type $\mathcal{L}_u \xi = 0$, $\nabla_u \xi = 0$), then they appear as "first integrals" of the deviation equation in the sense of solution of this equation.

Some conditions (part of which appear as solutions) for deviation equations obtained on the ground of these conditions and the generalized deviation identity are given below as generalizations for L_n-spaces:

1. $\mathcal{L}_\xi u = 0$, $\nabla_u u = a = 0$. Geodesic (auto-parallel) deviation equation (Jacobi equation), (Levi-Civita, 1925) [22]
$$\nabla_u \nabla_u \xi = [R(u, \xi)]u - \nabla_u[T(\xi, u)]. \tag{29}$$
2. $\mathcal{L}_\xi u = 0$, $\nabla_u u = a = ku$, $k \in C^r(M)$, $r \geq 2$. Jacobi equation in non-canonical form
$$\nabla_u \nabla_u \xi = [R(u, \xi)]u + (\xi k)u + k\nabla_\xi u - \nabla_u[T(\xi, u)]. \tag{30}$$
3. $\mathcal{L}_\xi u = 0$. Deviation equation of Synge and Schild (deviation equation without dragging-along) (Synge and Schild, 1956) (Schmutzer, 1968)
$$\nabla_u \nabla_u \xi = [R(u, \xi)]u + \nabla_\xi a - \nabla_u[T(\xi, u)]. \tag{31}$$
4. $\mathcal{L}_\xi u = fu$, $f \in C^r(M)$, $r \geq 2$. Deviation equation with co-linear dragging-along (Mashhoon, 1975) (Hawking and Ellis, 1973)
$$\nabla_u \nabla_u \xi = [R(u, \xi)]u + \nabla_\xi a - (uf)u - 2fa - \nabla_u[T(\xi, u)], \tag{32}$$
5. $\mathcal{L}_\xi u = fu$, $f = 1/a - 1$, $a \in C^r(M)$, $r \geq 2$. Deviation equation with co-linear dragging-along
$$\nabla_u \nabla_u \xi = [R(u, \xi)]u + \nabla_\xi a - 2\left(\frac{1}{a} - 1\right)a - \left[u\left(\frac{1}{a}\right)\right]u - \nabla_u[T(\xi, u)]. \tag{33}$$
6. $\mathcal{L}_\xi u = fu$, $\mathcal{L}_\xi a = 2fa + (uf)u$. Deviation equation of co-linear dragging-along of u and a (see Jacobi equation No.(1))
$$\nabla_u \nabla_u \xi = [R(u, \xi)]u - \nabla_u[T(\xi, u)].$$

7. $\mathcal{L}_\xi u = fu + g\xi$, $f, g \in C^r(M)$, $r \geq 2$. Deviation equation of dragging-along with friction

$$\nabla_u \nabla_u \xi = [R(u, \xi)]u + \nabla_\xi a - 2fa - (uf)u - (ug)\xi - g(\nabla_u \xi + \nabla_\xi u) - \nabla_u [T(\xi, u)]. \quad (34)$$

8. $\mathcal{L}_\xi u = \nabla_\xi u - T(\xi, u)$, $(\nabla_u \xi - 0)$. Deviation equation of parallel transport of ξ along u:

$$\nabla_u \nabla_u \xi = 0. \quad (35)$$

9. $\mathcal{L}_\xi u = (1/\alpha)[\nabla_u \xi + (1 - \alpha)u]$, $\alpha \in C^r(M)$, $r \geq 2$. Deviation equation of relative dragging-along (Iliev, 1980)

$$\nabla_u \nabla_u \xi = \frac{\alpha}{\alpha + 1} \left\{ [R(u, \xi)]u + \nabla_\xi a - 2\left(\frac{1}{\alpha} - 1\right) a - \left[u\left(\frac{1}{\alpha}\right)\right](u + \nabla_u \xi) - \frac{1}{\alpha} \nabla_{\nabla_u \xi} u - \nabla_u [T(\xi, u)] \right\}. \quad (36)$$

10. $\mathcal{L}_\xi u = qa$, $q \in C^r(M)$, $r \geq 2$. Deviation equation of abating dragging-along

$$\nabla_u \nabla_u \xi = [R(u, \xi)]u + \nabla_\xi a - (uq)a - q(\nabla_u a + \nabla_a u) - \nabla_u [T(\xi, u)]. \quad (37)$$

11. $\mathcal{L}_\xi u = -T(\xi, u)$. Deviation equation of dragging-along with torsion

$$\nabla_u \nabla_u \xi = [R(u, \xi)]u + \nabla_\xi a + \nabla_{T(\xi,u)} u. \quad (38)$$

12. $\nabla_u u = a = 0$. Deviation equation of auto-parallel transport of u:

$$\nabla_u \nabla_u \xi = [R(u, \xi)]u - \nabla_u [\mathcal{L}_\xi u + T(\xi, u)] - \nabla_{\mathcal{L}_\xi u} u. \quad (39)$$

13. $\nabla_\xi u = 0$. Deviation equation of parallel transport of u along ξ:

$$\nabla_u \nabla_u \xi = -\nabla_u [\mathcal{L}_\xi u + T(\xi, u)]. \quad (40)$$

14. $\mathcal{L}_\xi u = 0$, $\nabla_\xi u = 0$. Deviation equation without dragging-along ξ and with parallel transport of u along ξ:

$$\nabla_u \nabla_u \xi = -\nabla_u [T(\xi, u)]. \quad (41)$$

15. $\nabla_\xi u = 0$, $\nabla_u u = a = 0$. Deviation equation with parallel transport of u along ξ and u (see No. 13)

$$\nabla_u \nabla_u \xi = -\nabla_u [\mathcal{L}_\xi u + T(\xi, u)]. \quad (42)$$

*** The deviation equations of Synge and Schild (see No. 3) are form invariant with respect to different conditions required for $\mathcal{L}_\xi a$. The same is also valid for the generalized deviation identity written in the form

$$\nabla_u \nabla_u \xi \equiv [R(u, \xi)]u + \nabla_\xi a - \nabla_u [\mathcal{L}_\xi u + T(\xi, u)] - \nabla_{\mathcal{L}_\xi u} u.$$

In cases when deviation equations are obtained only by means of conditions of a type of "first integrals" for the contravariant vector field ξ, the corresponding deviation equations are valid for arbitrary vector field u. The conditions appear as sufficient but not necessary conditions for the existence of the corresponding deviation equations. Other sufficient conditions as "first integrals" can also be found for one and the same type of deviation equation.

Proposition 1. *The necessary and sufficient conditions for the existence of the deviation equation of Synge and Schild in L_n-spaces*

$$\nabla_u \nabla_u \xi \equiv [R(u, \xi)]u + \nabla_\xi a - \nabla_u [T(\xi, u)]$$

follow from the generalized deviation identity and can be written in the form

$$\mathcal{L}_\xi a = [\mathcal{L}\Gamma(\xi, u)]u, \qquad \mathcal{L}_\xi a^i = u^k u^l \mathcal{L}_\xi \Gamma^i_{kl}. \tag{43}$$

Proposition 2. *The necessary and sufficient conditions for the existence of the auto-parallel (geodesic) deviation equation (Jacobi equation) in L_n-spaces*

$$\nabla_u \nabla_u \xi \equiv [R(u, \xi)]u - \nabla_u [T(\xi, u)]$$

are the conditions

$$\mathcal{L}_\xi a = \nabla_\xi a + [\mathcal{L}\Gamma(\xi, u)]u, \qquad \mathcal{L}_\xi a^i = a^i{}_{;j} \xi^j + u^k u^l \mathcal{L}_\xi \Gamma^i_{kl}. \tag{44}$$

The last two formulated propositions can be proved in trivial manner by means of the generalized deviation identity.

It follows from the examples 1 and 6 that for deviation equations, the Jacobi (auto-parallel, geodesic deviation) equation allows other conditions than the condition $\nabla_u u = a = 0$, considered until now in the literature. Analogous statement is also valid for the deviation equation of Synge and Schild. The deviation equation of Synge and Schild can be considered as a corollary of the equation $\mathcal{L}_\xi u = 0$ ($\mathcal{L}_u \xi = 0$) for a vector field ξ and an arbitrary given vector field u. The last equation appears only as a sufficient but not as a necessary condition for the existence of the deviation equation of Synge and Schild, which, therefore, allows other "first integrals" as well. Let us argue this fact in more detail.

The way of getting the deviation equation of Synge and Schild gives the possibility for proving the following proposition:

Proposition 3. *Every vector field ξ which satisfies the equation $\mathcal{L}_\xi u = 0$ ($\mathcal{L}_\xi u^i = 0$) for an arbitrary (given) vector field u is a solution of the deviation equation of Synge and Schild.*

Proof. There are at least two ways [34,37] for proving this proposition:

1. The proof follows immediately from the generalized identity and the condition $\mathcal{L}_\xi u = 0$.
2. From the condition $\mathcal{L}_\xi u = 0$ and after covariant differentiation along u of the expression for $\nabla_u \xi$ the deviation equation follows. □

Corollary. *The condition $\mathcal{L}_\xi u = 0$ is a "first integral" for the deviation equation of Synge and Schild [for an arbitrary (given) vector field u].*

Remark. Under "first integral" here one can define a quantity whose covariant derivative along an arbitrary vector field u leads to the deviation equation of a concrete type (here of Synge and Schild).

Remark. In finding out deviation equations, different authors used only sufficient (or "first integrals") conditions for these equations (like those in Proposition 3). They do not take into account that the obtained equations can also fulfill other sufficient conditions than the considered one (see for example, [34,37]).

If we introduce a metric in a L_n-space leading to a (L_n, g)-space then the metric could come into use. The second covariant derivative of a vector field ξ along a non-null (non-isotropic) vector field u can be written in two parts: one is collinear to u, and the other is orthogonal to the vector field u. The second term can be interpreted as a relative acceleration between two points, lying on a hyper-surface orthogonal to the vector field u. Since the (infinitesimal) deviation vector has also to lie on this hyper-surface, then in this case ξ has to obey the condition

$$g(\xi, u) = 0, \tag{45}$$

or ξ has to be in the form

$$\begin{aligned} &\xi_\perp = \bar{g}[h_u(\xi)] = g^{ik} h_{kl} \xi^l e_i, \quad g(\xi_\perp, u) = 0, \quad \bar{g} = g^{ij} e_i e_j, \\ &h_u = g - \frac{1}{e} g(u) \otimes g(u) = h_{ij} e^i e^j, \qquad e_i e_j = \frac{1}{2}(e_i \otimes e_j + e_j \otimes e_i), \\ &e^i e^j = \frac{1}{2}(e^i \otimes e^j + e^j \otimes e^i), \qquad g = g_{ij} e^i e^j, \qquad e = g(u,u) \neq 0, \\ &g_{ij} = g_{ji}, \qquad e^i = \mathrm{d}x^i \quad \text{(in a co-ordinate basis)}. \end{aligned} \tag{46}$$

Definition 2. The deviation equation which is obtained for

$$\bar{g}[h_u(\nabla_u \nabla_u \xi)] \ \text{ or for } \ h_u(\nabla_u \nabla_u \xi)$$

under the conditions

$$\mathcal{L}_{\xi_\perp} u = 0, \quad g(u, \xi_\perp) = 0, \quad \xi_\perp = \bar{g}[h_u(\xi)], \tag{47}$$

is called *projective deviation equation of Synge and Schild* [42].

It follows from the form of $\bar{g}[h_u(\nabla_u \nabla_u \xi)]$ that this equation can be written in the form [40]

$$\bar{g}[h_u(\nabla_u \nabla_u \xi_\perp)] = \bar{g}[A(\xi_\perp)] = \bar{g}[{}_s D(\xi_\perp)] + \bar{g}[W(\xi_\perp)] + \frac{1}{n-1} U \xi_\perp, \tag{48}$$

or in index form

$$g^{ij} h_{jk} (\xi^k_{\perp;l} u^l)_{;m} u^m = g^{ij} A_{jk} \xi^k_\perp = g^{ij} ({}_s D_{jk} + W_{jk}) \xi^k_\perp + \frac{1}{n-1} U \xi^i_\perp, \tag{49}$$

where $\xi^k_\perp = g^{kl} h_{lm} \xi^m$, $h_u(\xi_\perp) = h_u(\bar{g}) h_u(\xi) = h_u(\xi)$, $\bar{g}[h_u(\xi_\perp)] = \bar{g}[h_u(\xi)] = \xi_\perp$.

The projective deviation equation can also be written in an equivalent form

$$h_u(\nabla_u \nabla_u \xi_\perp) = {}_s D(\xi_\perp) + W(\xi_\perp) + \frac{1}{n-1} U g(\xi_\perp). \tag{50}$$

Every vector field $\xi_\perp$ [for an arbitrary non-null (non-isotropic) vector field u] which fulfills the conditions $\mathcal{L}_{\xi_\perp} u = 0$, $\xi_\perp = \bar{g}[h_u(\xi)]$, is a solution of the projective deviation equation of Synge and Schild. In other words, the solution of the equation $\mathcal{L}_{\xi_\perp} u = 0$ (or $\mathcal{L}_u \xi_\perp = 0$) for a vector field $\xi_\perp(x^k)$ and a given vector field $u(x^k)$ is a solution of the projective deviation equation. It follows in this case that if the components of the vector field $\xi = \xi^i e_i = \xi^k \partial_k$ should be solutions of a homogeneous (or non-homogeneous) oscillator equation, then an additional equation for the vector field u has to be proposed, which could lead to such properties of ξ.

Deviation equations can be considered in special cases of U_n-spaces [(pseudo) Riemannian spaces with torsion] or in V_n-spaces. In spaces with affine connection and metric [(L_n, g)-spaces] the generalized deviation identity can be connected with the kinematic characteristics of the relative acceleration (such as shear acceleration, rotation acceleration and expansion acceleration) and with the kinematic characteristics of the relative velocity (shear, rotation, expansion).

Remark. The action of the deviation operator on covariant vector fields is determined by its structure and especially by the Lie differential operator. For a covariant vector field there is no appropriate deviation identity in L_n-spaces for describing the second covariant derivative of this field.

4. Conclusions

Deviation operator, constructed by means of the covariant and the Lie-differential operator, can be used for obtaining and investigating deviation equations in L_n-spaces in an analogous way as in (L_n, g)-spaces. The situation changes when metric tensor fields are used to connect the deviation equations with the kinematic characteristics of contravariant vector fields. The probability of arising problems would be mostly connected with the physical interpretation of the introduced quantities than with the differential-geometric structure of a L_n-space.

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